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Real Roots of Real Cubics and Optimization

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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The solution of the cubic equation has a century-long history; however, the usual presentation is geared towards applications in algebra and is somewhat inconvenient to use in optimization where frequently the main interest lies in real roots. In this paper, we first present the roots of the cubic in a form that makes them convenient to use and we also focus on information on the location of the real roots. Armed with this, we provide several applications in convex analysis and optimization where we compute Fenchel conjugates, proximal mappings and projections.

Keywords: Convex quartic, cubic equation, cubic polynomial, Fenchel conjugate, projection, proximal mapping, root.

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1. Introduction

The history of solving cubic equations is rich and centuries old; see, e.g., Confalonieri's recent book [11] on Cardano's work. Cubics do also appear in convex and nonconvex optimization. However, treatises on solving the cubic often focus on the general complex case making the results less useful to optimizers. More precisely, entering a *symbolic* cubic into SageMath [21] or WolframAlpha [22] yields long unwieldy formulas where one root is real when the coefficients are all real and the others look nonreal due to the presence of $\sqrt{-1}$. However, these packages[1] give little insight about exactly which roots are actually real and how the real roots relate to each other in terms of the ordering of the real line.

The purpose of this paper is two-fold. We present a largely self-contained derivation of the solution of the cubic with an emphasis on usefulness to practitioners by identifying the real roots and pointing out their ordering. We do not claim novelty of

[1]The same comment likely applies to commercial packages such as Mathematica and Maple. In this paper, we do not require that the reader has access to costly commercial software.

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these results; however, the presentation appears to be particularly convenient. We then turn to novel results in convex analysis (see Rockafellar's seminal book [20] for background material). We show how the formulas can be used to compute Fenchel conjugates and proximal mappings of some convex functions. We also discuss projections on convex and nonconvex sets. There is an obvious need and broad appeal for these results; in fact, a preprint of this paper was already cited by engineers in their work on power spectral density estimation (see [15, Appendix A]) and also by physicists in their work on two-qubit states [23, Appendix E]. Recently, the real root characterizations of cubic polynomials have also been used to describe the geometry of the function space represented by a linear convolutional network in deep learning [14] and to establish bounds in federated learning [13].

The paper is organized as follows. In Section 2, we collect some facts on polynomials. Section 3 contains a self-contained treatment of the depressed cubic; in turn, this leads quickly to counterparts for the general cubic in Section 4. Section 5 concerns convex quartics – we compute their Fenchel conjugates and proximal mappings. In Section 6, we present a formula for the proximal mapping of the convex reciprocal function. An explicit formula for the projection onto the epigraph of a parabola is provided in Section 7. In Section 8, we derive a formula for the projection of certain points onto a rectangular hyperbolic paraboloid. In Section 9, we revisit the proximal mapping of the closure of a perspective function. Finally, the appendices contain the proofs of the results we presented.

2. Some facts on polynomials

We now collect some properties of polynomials that are well known; as a reference, we recommend [19].

Fact 2.1. *Let $f(x)$ be a nonconstant complex polynomial and let $r \in \mathbb{C}$ such that $f(r) = 0$. Then the multiplicity of r is is the smallest integer k such that the kth derivative at r is nonzero: $f^{(k-1)}(r) = 0$ and $f^{(k)}(r) \neq 0$. When $k = 1$, 2, or 3, then we say that r is a simple, double, or triple root, respectively.*

Fact 2.2. (Vieta) *Suppose $f(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial (i.e., $a \neq 0$) with complex coefficients. If r_1, r_2, r_3 denote the (possibly repeated and complex) roots of f, then*

$$r_1 + r_2 + r_3 = -\frac{b}{a} \tag{1a}$$

$$r_1r_2 + r_1r_3 + r_2r_3 = \frac{c}{a} \tag{1b}$$

$$r_1r_2r_3 = -\frac{d}{a}. \tag{1c}$$

Conversely, if r_1, r_2, r_3 in $\mathbb{C}$ satisfy (1)*, then they are the (possibly repeated) roots of f.*

Fact 2.3. *Suppose $f(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial (i.e., $a \neq 0$) with real coefficients. Then f has three (possibly complex) roots (counting multiplicity). More precisely, exactly one of the following holds:*

(i) *f has exactly one real root which either is simple (and the two remaining roots are nonreal simple roots and conjugate to each other) or is a triple root.*

(ii) *f has exactly two distinct real roots: one is simple and the other double.*

(iii) *f has exactly three distinct simple real roots.*

Remark 2.4. We mention that the roots of a polynomial of a *fixed* degree depend continuously on the coefficients – see [19, Theorem 1.3.1] for a precise statement and also the other results in [19, Section 1.3].

3. The depressed cubic

In this section, we study the *depressed* cubic

$$g(z) := z^3 + pz + q, \quad \text{where } p \in \mathbb{R} \text{ and } q \in \mathbb{R}. \tag{2}$$

For the reader's convenience, we relegate the proofs to the appendix.

Theorem 3.1. *Because*

$$g'(z) = 3z^2 + p \text{ and } g''(z) = 6z, \tag{3}$$

we see that 0 is the only inflection point of g: g is strictly concave on $\mathbb{R}_-$ and g is strictly convex on $\mathbb{R}_+$.

Moreover, exactly one of the following cases occurs:

(i) *$p < 0$: Set $z_\pm := \pm\sqrt{-p/3}$. Then $z_- < z_+$, $z_\pm$ are two distinct simple roots of g', g is strictly increasing on $]-\infty, z_-]$, g is strictly decreasing on $[z_-, z_+]$, g is strictly increasing on $[z_+, +\infty[$. Moreover,*

$$g(z_-)g(z_+) = 4\Delta, \quad \text{where } \Delta := (p/3)^3 + (q/2)^2, \tag{4}$$

and this case trifurcates further as follows:

(a) *$\Delta > 0$: Then g has exactly one real root r. It is simple and given by*

$$r := u_- + u_+, \quad \text{where } u_\pm := \sqrt[3]{\frac{-q}{2} \pm \sqrt{\Delta}}. \tag{5}$$

The two remaining simple nonreal roots are

$$-\tfrac{1}{2}(u_- + u_+) \pm \mathrm{i}\tfrac{1}{2}\sqrt{3}(u_- - u_+). \tag{6}$$

(b) *$\Delta = 0$: If $q > 0$ (resp. $q < 0$), then $2z_-$ (resp. $2z_+$) is a simple real root while z_+ (resp. z_-) is a double root. Moreover, these cases can be combined into*[2]

$$\begin{aligned} \frac{3q}{p} &= 2\sqrt[3]{\frac{-q}{2}} \text{ is a simple root of } g \text{ and} \\ \frac{-3q}{2p} &= -\sqrt[3]{\frac{-q}{2}} \text{ is a double root of } g. \end{aligned} \tag{7}$$

(c) *$\Delta < 0$: Then g has three simple real roots r_-, r_0, r_+ satisfying the relations $r_- < z_- < r_0 < z_+ < r_+$. Indeed, set*

$$\theta := \arccos \frac{-q/2}{(-p/3)^{3/2}}, \tag{8}$$

[2] Observe that this is the case when $\Delta \to 0^+$ in (i)(a)

which lies in $]0,\pi[$*, and then define* z_0, z_1, z_2 *by*

$$z_k := 2(-p/3)^{1/2} \cos\left(\frac{\theta + 2k\pi}{3}\right). \tag{9}$$

Then $r_- = z_1$*,* $r_0 = z_2$*, and* $r_+ = z_0$*.*

(ii) $p = 0$*: Then* g' *has a double root at* 0*, and* g *is strictly increasing on* $\mathbb{R}$*. The only real root is*

$$r := (-q)^{1/3}. \tag{10}$$

If $q = 0$*, then* r *is a triple root. If* $q \neq 0$*, then* r *is a simple root and the remaining nonreal simple roots are* $-\frac{1}{2}r \pm \mathrm{i}\frac{1}{2}\sqrt{3}r$*.*

(iii) $p > 0$*: Then* g' *has no real root,* g *is strictly increasing on* $\mathbb{R}$*, and* g *has exactly one real root* r*. It is simple and given by*

$$r := u_- + u_+, \quad \text{where } u_\pm := \sqrt[3]{\frac{-q}{2} \pm \sqrt{\Delta}} \text{ and } \Delta := (p/3)^3 + (q/2)^2. \tag{11}$$

Once again, the two remaining simple nonreal roots are

$$-\tfrac{1}{2}(u_- + u_+) \pm \mathrm{i}\tfrac{1}{2}\sqrt{3}(u_- - u_+). \tag{12}$$

Proof. See Appendix A. □

We now provide a concise version of Theorem 3.1:

Corollary 3.2. (Trichotomy) *Set* $\Delta := (p/3)^3 + (q/2)^2$*. Then exactly one of the following holds:*

(i) $p = 0$ *or* $\Delta > 0$*: Then* g *has exactly one real root and it is given by*

$$\sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}. \tag{13}$$

(ii) $p < 0$ *and* $\Delta = 0$*: Then* g *has exactly two real roots which are given by*

$$\frac{3q}{p} = 2\sqrt[3]{\frac{-q}{2}} \quad \text{and} \quad \frac{-3q}{2p} = -\sqrt[3]{\frac{-q}{2}}. \tag{14}$$

(iii) $\Delta < 0$*: Then* g *has exactly three real roots* z_0, z_1, z_2 *which are given by*

$$z_k := 2(-p/3)^{1/2} \cos\left(\frac{\theta + 2k\pi}{3}\right), \quad \text{where } \theta := \arccos \frac{-q/2}{(-p/3)^{3/2}}, \tag{15}$$

and where $z_1 < z_2 < z_0$*.*

As an illustration, we present an example that follows immediately from Corollary 3.2 (with $-q = p > 0$ and thus $\Delta > 0$):

Example 3.3. Suppose that $p > 0$. Then

$$z^3 + pz - p \tag{16}$$

has exactly one real root which is given by

$$\sqrt[3]{\frac{p}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{p}{2}\right)^2}} + \sqrt[3]{\frac{p}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{p}{2}\right)^2}}. \tag{17}$$

Remark 3.4. Example 3.3 is immediately useful to derive an explicit formula of a Bregman proximal mapping whose existence was announced in [7, Proposition 5.1] in the paper by Bolte, Sabach, Teboulle, and Vaisbourd (see also [12, Proposition 2.8] in the paper by Godeme, Fadili, Buet, Zerrad, Lequime, and Amra).

4. The general cubic

In this section, we turn to the general cubic

$$f(x) := ax^3 + bx^2 + cx + d, \quad \text{where } a, b, c, d \text{ are in } \mathbb{R} \text{ and } a > 0. \tag{18}$$

(The case $a < 0$ is treated similarly.) Note that $f''(x) = 6ax + 2b$ has exactly one zero, namely

$$x_0 := \frac{-b}{3a}. \tag{19}$$

The change of variables

$$x = z + x_0 \tag{20}$$

leads to the well known depressed cubic

$$g(z) := z^3 + pz + q, \quad \text{where } p := \frac{3ac - b^2}{3a^2} \text{ and } q := \frac{27a^2d + 2b^3 - 9abc}{27a^3} \tag{21}$$

which we reviewed in Section 3. Here $ag(z) = f(x) = f(z + x_0)$ so the roots of g are precisely those of f, translated by x_0:

$$x \text{ is a root of } f \Leftrightarrow x - x_0 \text{ is a root of } g. \tag{22}$$

So all we need to do is find the roots of g, and then add x_0 to them, to obtain the roots of f. Because the change of variables (20) is linear, multiplicity of the roots are preserved. Translating some of the results from Theorem 3.1 for g to f gives the following:

Theorem 4.1. *f is strictly concave on $]-\infty, x_0]$ and is strictly convex on $[x_0, +\infty[$, where x_0 is the unique inflection point of f defined in* (19)*. Recall the definitions of p, q from* (21) *and also set*

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{(3ac - b^2)^3}{(9a^2)^3} + \frac{(27a^2d + 2b^3 - 9abc)^2}{(54a^3)^2}. \tag{23}$$

Then exactly one of the following cases occurs:

(i) $\boxed{b^2 > 3ac \Leftrightarrow p < 0}$*: Set $x_\pm := (-b \pm \sqrt{b^2 - 3ac})/(3a)$. Then $x_\pm$ are two distinct simple roots of f', f is strictly increasing on $]-\infty, x_-]$, f is strictly decreasing on $[x_-, x_+]$, f is strictly increasing on $[x_+, +\infty[$. This case trifurcates further as follows:*

(a) $\boxed{\Delta > 0}$*: Then f has exactly one real root; moreover, it is simple and given by*

$$x_0 + u_- + u_+, \quad \textit{where } u_\pm := \sqrt[3]{\frac{-q}{2} \pm \sqrt{\Delta}}. \tag{24}$$

The two remaining simple nonreal roots are

$$x_0 - \tfrac{1}{2}(u_- + u_+) \pm \mathrm{i}\tfrac{1}{2}\sqrt{3}(u_- - u_+).$$

(b) $\boxed{\Delta = 0}$: *Then f has two distinct real roots: The simple root is*

$$x_0 + \frac{3q}{p} = x_0 + 2\sqrt[3]{\frac{-q}{2}} = \frac{4abc - b^3 - 9a^2 d}{a(b^2 - 3ac)} \tag{25}$$

and the double root is

$$x_0 - \frac{3q}{2p} = x_0 - \sqrt[3]{\frac{-q}{2}} = \frac{9ad - bc}{2(b^2 - 3ac)}. \tag{26}$$

(c) $\boxed{\Delta < 0}$: *Then f has three simple real roots r_-, r_0, r_+ satisfying the relations $r_- < x_- < r_0 < x_+ < r_+$. Indeed, set*

$$\theta := \arccos \frac{-q/2}{(-p/3)^{3/2}}, \tag{27}$$

which lies in $]0, \pi[$, and then define y_0, y_1, y_2 by

$$y_k := x_0 + 2(-p/3)^{1/2} \cos\left(\frac{\theta + 2k\pi}{3}\right). \tag{28}$$

Then $r_- = y_1$, $r_0 = y_2$, and $r_+ = y_0$.

(ii) $\boxed{b^2 = 3ac \Leftrightarrow p = 0}$: *Then f is strictly increasing on $\mathbb{R}$ and its only real root is*

$$r := x_0 + (-q)^{1/3}. \tag{29}$$

If $q = 0$, then r is a triple root. If $q \neq 0$, then r is a simple root and the remaining nonreal simple roots are $x_0 - \frac{1}{2}(-q)^{1/3} \pm \mathrm{i}\frac{1}{2}\sqrt{3}(-q)^{1/3}$.

(iii) $\boxed{b^2 < 3ac \Leftrightarrow p > 0}$: *Then f is strictly increasing on $\mathbb{R}$, and f has exactly one real root; moreover, it is simple and given by*

$$x_0 + u_- + u_+, \quad \textit{where } u_\pm := \sqrt[3]{\frac{-q}{2} \pm \sqrt{\Delta}}. \tag{30}$$

The two remaining simple nonreal roots are $x_0 - \frac{1}{2}(u_- + u_+) \pm \mathrm{i}\frac{1}{2}\sqrt{3}(u_- - u_+)$.

Remark 4.2. Of course, descriptions for the roots of the cubic are well known; see, e.g., [1, Section 3.8.2]. However, none of the works we are aware of provides an easy-to-use description of the roots and their ordering when real.

Considering the case when $a = 1$, $b = -\alpha$, $c = 0$, and $d = -\gamma$, followed by some simplifications, we obtain the following:

Example 4.3. Let $\alpha > 0$ and $\gamma > 0$. Then

$$z^3 - \alpha z^2 - \gamma \tag{31}$$

has exactly one real root which is given by

$$\frac{\alpha}{3} + \sqrt[3]{\left(\frac{\alpha}{3}\right)^3 + \frac{\gamma + \sqrt{\delta}}{2}} + \sqrt[3]{\left(\frac{\alpha}{3}\right)^3 + \frac{\gamma - \sqrt{\delta}}{2}}, \quad \text{where } \delta := \gamma^2 + \frac{4}{27}\alpha^3\gamma > 0. \tag{32}$$

Remark 4.4. Example 4.3 appears in work by Ahookhosh, Hien, Gillis, and Patrinos on Bregman proximal alternating linearized minimization; see [2, Proof of Theorem 5.2].

In turn, Corollary 3.2 turns into

Corollary 4.5. *Recall* (19) *and* (21), *and set*

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{(3ac-b^2)^3}{(9a^2)^3} + \frac{(27a^2d+2b^3-9abc)^2}{(54a^3)^2} \tag{33}$$

Then exactly one of the following holds:

(i) $\boxed{b^2 = 3ac \text{ or } \Delta > 0}$: *Then f has exactly one real root and it is given by*

$$x_0 + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}. \tag{34}$$

(ii) $\boxed{b^2 > 3ac \text{ and } \Delta = 0}$: *Then f has exactly two real roots which are given by*

$$x_0 + \frac{3q}{p} = x_0 + 2\sqrt[3]{\frac{-q}{2}} \text{ and } x_0 + \frac{-3q}{2p} = x_0 - \sqrt[3]{\frac{-q}{2}}. \tag{35}$$

(iii) $\boxed{\Delta < 0}$: *Then f has exactly three real (simple) roots r_0, r_1, r_2, where*

$$r_k := x_0 + 2(-p/3)^{1/2}\cos\left(\frac{\theta + 2k\pi}{3}\right), \qquad \theta := \arccos\frac{-q/2}{(-p/3)^{3/2}}, \tag{36}$$

and $r_1 < r_2 < r_0$.

5. Convex analysis of the general quartic

In this section, we study the function

$$h(x) := \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon, \quad \text{where } \alpha, \beta, \gamma, \delta, \varepsilon \text{ are in } \mathbb{R} \text{ with } \alpha \neq 0. \tag{37}$$

We start by characterizing convexity.

Proposition 5.1. (Convexity) *The general quartic* (37) *is convex if and only if*

$$\alpha > 0 \text{ and } 8\alpha\gamma \geq 3\beta^2. \tag{38}$$

Proof. See Appendix B.1. □

We shall now assume the above characterization of convexity for the remainder of this section:

$$h \text{ is convex, i.e., } \alpha > 0 \text{ and } 8\alpha\gamma \geq 3\beta^2. \tag{39}$$

We now turn to the the computation of the Fenchel conjugate (see [20, Section 12]).

Proposition 5.2. (Fenchel conjugate) *Recall our assumptions* (37) *and* (39). *Let $y \in \mathbb{R}$. Then*

$$h^*(y) = yx_y - h(x_y), \tag{40}$$

where $p := (8\alpha\gamma - 3\beta^2)/(16\alpha^2) \geq 0$, $q := (8\alpha^2(\delta - y) + \beta^3 - 4\alpha\beta\gamma)/(32\alpha^3)$, $\Delta := (p/3)^2 + (q/2)^2 \geq 0$, *and*

$$x_y := -\frac{\beta}{4\alpha} + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}. \tag{41}$$

Proof. See Appendix B.2. □

Next, we compute the proximal mapping (see [18] and also [20, Section 31]).

Proposition 5.3. (Proximal mapping) *Recall our assumptions* (37) *and* (39). *Let* $y \in \mathbb{R}$. *Then*

$$\operatorname{Prox}_h(y) = -\frac{\beta}{4\alpha} + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}, \tag{42}$$

where

$$p := \frac{4\alpha(1+2\gamma) - 3\beta^2}{16\alpha^2}, \quad q := \frac{8\alpha^2(\delta - y) + \beta^3 - 2\alpha\beta(1+2\gamma)}{32\alpha^3}, \tag{43}$$

and $\Delta := (p/3)^3 + (q/2)^2 \geq 0$.

Proof. See Appendix B.3. □

Example 5.4. Suppose that

$$h(x) = x^4 + x^3 + x^2 + x + 1, \tag{44}$$

and let $y \in \mathbb{R}$. Then h is convex and

$$h^*(y) = yx_y - h(x_y), \quad \text{where} \tag{45a}$$

$$x_y = -\frac{1}{4} + \frac{1}{2}\sqrt[3]{y - \frac{5}{8} + \sqrt{(y - \frac{5}{8})^2 + (\frac{5}{4})^3}} + \frac{1}{2}\sqrt[3]{y - \frac{5}{8} - \sqrt{(y - \frac{5}{8})^2 + (\frac{5}{4})^3}}. \tag{45b}$$

Moreover,

$$\operatorname{Prox}_h(y)$$

$$= -\frac{1}{4} + \frac{1}{2}\sqrt[3]{y - \frac{3}{8} + \sqrt{(y - \frac{3}{8})^2 + (\frac{3}{4})^3}} + \frac{1}{2}\sqrt[3]{y - \frac{3}{8} - \sqrt{(y - \frac{3}{8})^2 + (\frac{3}{4})^3}}. \tag{46}$$

See Figure 1 for a visualization.

Proof. See Appendix B.4. □

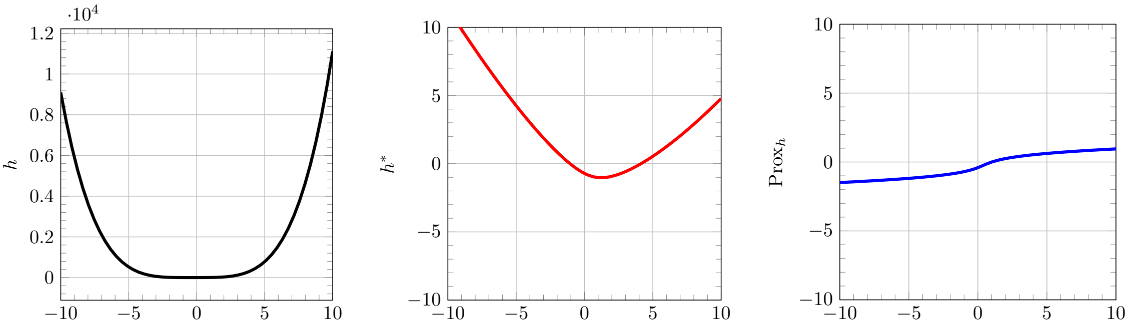

Figure 1: A visualization of Example 5.4. Depicted are h (left), its conjugate h^* (middle), and the proximal mapping Prox_h (right).

Example 5.5. Suppose that

$$h(x) = \alpha x^4, \quad \text{where } \alpha > 0, \tag{47}$$

and let $y \in \mathbb{R}$. Then

$$h^*(y) = \frac{3}{4(4\alpha)^{1/3}} y^{4/3} \tag{48}$$

and

$$\operatorname{Prox}_h(y) = \frac{1}{2}\sqrt[3]{\frac{y}{\alpha} + \sqrt{\frac{1 + 27\alpha y^2}{27\alpha^3}}} + \frac{1}{2}\sqrt[3]{\frac{y}{\alpha} - \sqrt{\frac{1 + 27\alpha y^2}{27\alpha^3}}}. \tag{49}$$

Proof. See Appendix B.5. □

Remark 5.6. The Fenchel conjugate formula (48) is known and can also be computed by combining, e.g., [3, Example 13.2(i) with Proposition 13.23(i)]. The prox formula (49) appears – with a typo though – in [3, Example 24.38(v)]. Finally, for a Maple[3] implementation for quartics, see Lucet's [17] and also the related papers [8] and [16] by Borwein and Hamilton and by Lauster, Luke, and Tam, respectively.

6. The proximal mapping of α/x

In this section, we study the convex reciprocal function

$$h(x) := \begin{cases} \alpha/x, & \text{if } x > 0; \\ +\infty, & \text{if } x \leq 0, \end{cases} \qquad \text{where } \alpha > 0. \tag{50}$$

The Fenchel conjugate h^*, which requires only solving a *quadratic* equation, is essentially known (e.g., combine [3, Example 13.2(ii) and Proposition 13.23(i)]), and given by

$$h^*(y) = \begin{cases} -2\sqrt{-\alpha y}, & \text{if } y \leq 0; \\ +\infty, & \text{if } y > 0. \end{cases} \tag{51}$$

The aim of this section is to explicitly compute Prox_h. The result is Proposition 6.1.

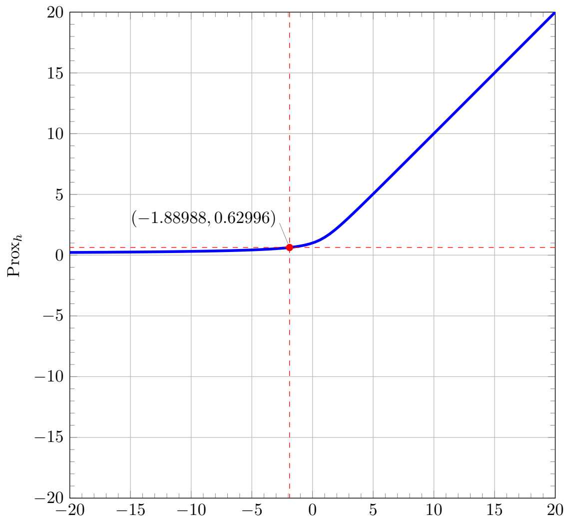

Figure 2: A visualization of Proposition 6.1 when $\alpha = 1$.

Proposition 6.1. *Suppose that h is given by (50), and let $y \in \mathbb{R}$. Now set $y_0 := -3\sqrt[3]{\alpha/4} \approx -1.88988\sqrt[3]{\alpha}$. Then we have the following three possibilities:*

(i) *If $y_0 < y$, then*

$$\operatorname{Prox}_h(y) = \frac{y}{3} + \sqrt[3]{\frac{\alpha}{2}+\left(\frac{y}{3}\right)^3+\sqrt{\alpha\left(\frac{\alpha}{4}+\left(\frac{y}{3}\right)^3\right)}} + \sqrt[3]{\frac{\alpha}{2}+\left(\frac{y}{3}\right)^3-\sqrt{\alpha\left(\frac{\alpha}{4}+\left(\frac{y}{3}\right)^3\right)}}.$$

(ii) *If $y = y_0$, then* $\operatorname{Prox}_h(y_0) = \sqrt[3]{\alpha}/\sqrt[3]{4} \approx 0.62996\sqrt[3]{\alpha}$.

(iii) *If $y < y_0$, then* $\operatorname{Prox}_h(y) = \dfrac{y}{3}\left(1 - 2\cos\left(\dfrac{1}{3}\arccos\dfrac{(y/3)^3+\alpha/2}{-(y/3)^3}\right)\right)$.

[3]Unfortunately, the commercial software Maple is not freely available.

Proof. See Appendix C. □

Remark 6.2. Suppose that $\alpha=1$. Then Prox_h was discussed in [9]; however, no explicit formulae were presented. For a visualization of Prox_h in this case, see Figure 2.

7. Projection onto the epigraph of a parabola

In this section, we study the projection onto the epigraph of the function

$$h\colon \mathbb{R}^n \to \mathbb{R}\colon \mathbf{x} \mapsto \alpha\|\mathbf{x}\|^2, \quad \text{where } \alpha > 0. \tag{52}$$

Theorem 7.1. *Set $E := \operatorname{epi} h \subseteq \mathbb{R}^{n+1}$. Let $(\mathbf{y},\eta) \in (\mathbb{R}^n \times \mathbb{R})$. If $(\mathbf{y},\eta) \in E$, then $P_E(\mathbf{y},\eta) = (\mathbf{y},\eta)$. So we assume that $(\mathbf{y},\eta) \in (\mathbb{R}^n \times \mathbb{R}) \smallsetminus E$, i.e., $\alpha\|\mathbf{y}\|^2 > \eta$. Set $\nu := \|\mathbf{y}\| \geq 0$,*

$$p := -\frac{(2\alpha\eta-1)^2}{12\alpha^2}, \quad q := \frac{(2\alpha\eta-1)^3 - 27\alpha^2\nu^2}{108\alpha^3}, \tag{53}$$

$\Delta := (p/3)^3 + (q/2)^2 = (27\alpha^2\nu^2 - 2(2\alpha\eta-1)^3)\nu^2/(1728\alpha^4)$, *and*

$$x := \begin{cases} -\dfrac{\alpha\eta+1}{3\alpha} + \sqrt[3]{-q/2+\sqrt{\Delta}} + \sqrt[3]{-q/2-\sqrt{\Delta}}, & \text{if } \Delta \geq 0; \\[2ex] -\dfrac{\alpha\eta+1}{3\alpha} + \dfrac{|2\alpha\eta-1|}{3\alpha}\cos\Big(\dfrac{1}{3}\arccos\dfrac{-q/2}{(-p/3)^{3/2}}\Big), & \text{if } \Delta < 0. \end{cases} \tag{54}$$

Then
$$P_E(\mathbf{y},\eta) = \Big(\frac{\mathbf{y}}{1+2\alpha x}, \eta + x\Big). \tag{55}$$

See Figure 3 on the next page for an illustration for the case $\alpha = 1/2$.

Proof. See Appendix D. □

8. On the projection of a rectangular hyperbolic paraboloid

In this section, X is a real Hilbert space and we set

$$S := \big\{(\mathbf{x},\mathbf{y},\gamma) \in X \times X \times \mathbb{R} \bigm| \langle \mathbf{x},\mathbf{y}\rangle = \alpha\gamma\big\}, \quad \text{where } \alpha \in \mathbb{R} \smallsetminus \{0\}. \tag{56}$$

Using the Hilbert product space norm $\|(\mathbf{x},\mathbf{y},\gamma)\| := \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \beta^2\gamma^2}$, where $\beta > 0$, we are interested in finding the projection onto S. Various cases were discussed in [4], but 3 were treated only implicitly. Armed with the cubic, we are now able to treat two of these cases explicitly (the remaining case features a quintic and remains hard). The first case concerns

$$P_S(\mathbf{z},-\mathbf{z},\gamma), \quad \text{when } \mathbf{z} \in X \smallsetminus \{0\} \text{ and } \alpha(\gamma - \alpha/\beta^2) < -\|\mathbf{z}\|^2/4, \tag{57}$$

while the second case is

$$P_S(\mathbf{z},\mathbf{z},\gamma), \quad \text{when } \mathbf{z} \in X \smallsetminus \{0\} \text{ and } \alpha(\gamma + \alpha/\beta^2) > \|\mathbf{z}\|^2/4. \tag{58}$$

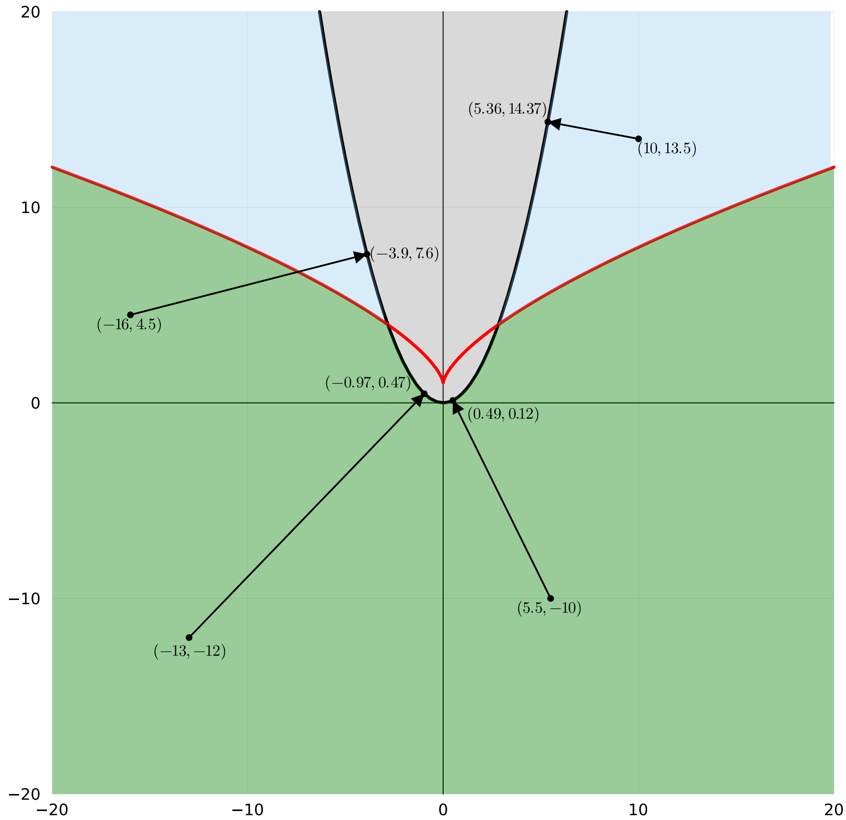

Figure 3: A visualization of Theorem 7.1 when $n = 1$ and $\alpha = 1/2$. The epigraph is shown in gray. The V-shaped curve corresponds to $\Delta = 0$, the dark gray region to $\Delta < 0$ (where trig functions are used) and the light gray region to $\Delta > 0$.

8.1. The case when (57) holds

Theorem 8.1. *Suppose* $\mathbf{z} \in X \smallsetminus \{0\}$, *set*

$$\zeta := \|\mathbf{z}\| > 0, \tag{59}$$

and assume that

$$\alpha(\gamma - \alpha/\beta^2) < -\zeta^2/4. \tag{60}$$

Set

$$p := -\frac{(\alpha+\beta^2\gamma)^2}{3\alpha^2}, \quad q := \frac{2(\alpha+\beta^2\gamma)^3}{27\alpha^3} + \frac{\beta^2\zeta^2}{\alpha^2}, \tag{61}$$

and

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{\beta^2\zeta^2}{\alpha^2}\Big(\frac{\beta^2\zeta^2}{4\alpha^2} + \frac{(\alpha+\beta^2\gamma)^3}{27\alpha^3}\Big). \tag{62}$$

If $\Delta \geq 0$, *then set*

$$x := \frac{2\alpha - \beta^2\gamma}{3\alpha} + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}; \tag{63}$$

and if $\Delta < 0$, *then set*

$$x := \frac{2\alpha - \beta^2\gamma}{3\alpha} + \delta\frac{2(\alpha+\beta^2\gamma)}{3\alpha}\cos\Big(\frac{1}{3}\Big((3+\delta)\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \tag{64a}$$

where

$$\delta := \operatorname{sign}(\alpha^2 + \alpha\beta^2\gamma) \in \{-1, 0, 1\}. \tag{64b}$$

Then $-1 < x < 1$ *and*

$$P_S(\mathbf{z}, -\mathbf{z}, \gamma) = \Big(\frac{\mathbf{z}}{1-x}, \frac{-\mathbf{z}}{1-x}, \gamma + \frac{\alpha x}{\beta^2}\Big). \tag{65}$$

Proof. See Appendix E.1. □

8.2. The case when (58) holds

Theorem 8.2. *Suppose* $\mathbf{z} \in X \smallsetminus \{0\}$, *set*

$$\zeta := \|\mathbf{z}\| > 0, \tag{66}$$

and assume that

$$\alpha(\gamma + \alpha/\beta^2) > \zeta^2/4. \tag{67}$$

Set

$$p := -\frac{(\beta^2\gamma - \alpha)^2}{3\alpha^2}, \quad q := \frac{2(\beta^2\gamma - \alpha)^3}{27\alpha^3} - \frac{\beta^2\zeta^2}{\alpha^2}, \tag{68}$$

and

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{\beta^2\zeta^2}{\alpha^2}\Big(\frac{\beta^2\zeta^2}{4\alpha^2} - \frac{(\beta^2\gamma - \alpha)^3}{27\alpha^3}\Big). \tag{69}$$

If $\Delta \geq 0$, *then set*

$$x := -\frac{2\alpha + \beta^2\gamma}{3\alpha} + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}; \tag{70}$$

and if $\Delta < 0$, *then set*

$$x := -\frac{2\alpha + \beta^2\gamma}{3\alpha} + \delta\frac{2(\alpha - \beta^2\gamma)}{3\alpha}\cos\Big(\frac{1}{3}\Big((2 + 2\delta)\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \tag{71a}$$

where

$$\delta := \operatorname{sign}(\alpha^2 - \alpha\beta^2\gamma) \in \{-1, 0, 1\}. \tag{71b}$$

Then $-1 < x < 1$ *and*

$$P_S(\mathbf{z}, \mathbf{z}, \gamma) = \Big(\frac{\mathbf{z}}{1+x}, \frac{\mathbf{z}}{1+x}, \gamma + \frac{\alpha x}{\beta^2}\Big). \tag{72}$$

Proof. See Appendix E.2. □

9. A proximal mapping of a closure of a perspective function

The following completes [3, Example 24.57] which stopped short of providing solutions for the cubic encountered there. For more on perspective functions, we refer the reader to Combettes's paper [10].

Example 9.1. Define the function h on $\mathbb{R}^n \times \mathbb{R}$ by

$$h(y, \eta) := \begin{cases} \|y\|^2/(2\eta), & \text{if } \eta > 0; \\ 0, & \text{if } y = 0 \text{ and } \eta = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{73}$$

Let $\gamma > 0$ and $(y, \eta) \in \mathbb{R}^n \times \mathbb{R}$. Then

$$\operatorname{Prox}_{\gamma h}(y, \eta) = \begin{cases} (0, 0), & \text{if } \|y\|^2 + 2\gamma\eta \leq 0; \\ \Big(\Big(1 - \frac{\gamma\lambda}{\|y\|}\Big)y,\ \eta + \frac{\gamma\lambda^2}{2}\Big), & \text{if } \|y\|^2 + 2\gamma\eta > 0, \end{cases} \tag{74}$$

where $p = 2(\eta + \gamma)/\gamma$, $\Delta = (p/3)^3 + (\|y\|/\gamma)^2$, and

$$\lambda = \begin{cases} \sqrt[3]{\frac{\|y\|}{\gamma} + \sqrt{\Delta}} + \sqrt[3]{\frac{\|y\|}{\gamma} - \sqrt{\Delta}}, & \text{if } \Delta \geq 0; \\ 2(-p/3)^{1/2}\cos\Big(\frac{1}{3}\arccos\frac{\|y\|/\gamma}{(-p/3)^{3/2}}\Big), & \text{if } \Delta < 0. \end{cases} \tag{75}$$

Proof. See Appendix F. □

Appendices

A. Proof of Theorem 3.1

Except for the formulas for the roots, all statements on g follow from calculus. For completeness, we include the proofs.

(i)(a): Because $\Delta > 0$ and g is strictly decreasing on $[z_-, z_+]$, it follows from (4) that g has the same sign on $[z_-, z_+]$ and so g has no root in that interval. Now g is strictly increasing on $]-\infty, z_-]$ and on $[z_+, +\infty[$; hence, g has exactly one real root r and it lies outside $[z_-, z_+]$. Note that r must be simple because the roots of g' are $z_\mp$ and $r \neq z_\mp$. Note that $u_- < u_+$. Next, $u_-^3 u_+^3 = (q/2)^2 - \Delta = -(p/3)^3$ and so

$$u_- u_+ = -p/3. \tag{76}$$

Also, $$u_-^3 + u_+^3 = \frac{-q}{2} - \sqrt{\Delta} + \frac{-q}{2} + \sqrt{\Delta} = -q. \tag{77}$$

Hence $$\begin{aligned} g(r) &= r^3 + pr + q \\ &= (u_- + u_+)^3 + p(u_- + u_+) + q \\ &= u_-^3 + u_+^3 + 3u_-u_+(u_- + u_+) + p(u_- + u_+) + q \\ &= \big(u_-^3 + u_+^3\big) + (3u_-u_+ + p)(u_- + u_+) + q \\ &= -q + \big(3(-p/3) + p\big)(u_- + u_+) + q \qquad \text{(using (76) and (77))} \\ &= 0 \end{aligned}$$

as claimed. Observe that we only need the properties (76) and (77) about u_-, u_+ to conclude that $u_- + u_+$ is a root of g. This observation leads us quickly to the two remaining complex roots: First, denote the primitive 3rd root of unity by ω, i.e.,

$$\omega := \exp(2\pi\mathrm{i}/3) = \cos(2\pi/3) + \mathrm{i}\sin(2\pi/3) = -\tfrac{1}{2} + \mathrm{i}\tfrac{1}{2}\sqrt{3}. \tag{78}$$

Then $\omega^2 = \overline{\omega} = -\frac{1}{2} - \mathrm{i}\frac{1}{2}\sqrt{3}$ and $\omega^3 = \overline{\omega}^3 = 1$. Now set

$$v_- := \omega u_- \quad \text{and} \quad v_+ := \omega^2 u_+ = \overline{\omega} u_+.$$

Then $v_- v_+ (\omega u_-) = (\omega^2 u_+) = \omega^3 u_- u_+ = u_- u_+ = -p/3$ by (76), and $v_-^3 + v_+^3 = (\omega u_-)^3 + (\omega^2 u_+)^3 = \omega^3 u_-^3 + \omega^6 u_+^3 = u_-^3 + u_+^3 = -q$ by (76). Hence

$$\begin{aligned} v_- + v_+ &= \omega u_- + \overline{\omega} u_+ \\ &= \big(-\tfrac{1}{2} + \mathrm{i}\tfrac{1}{2}\sqrt{3}\big)u_- + \big(-\tfrac{1}{2} - \mathrm{i}\tfrac{1}{2}\sqrt{3}\big)u_+ \\ &= -\tfrac{1}{2}(u_- + u_+) + \mathrm{i}\tfrac{1}{2}\sqrt{3}(u_- - u_+) \end{aligned}$$

and its conjugate are the remaining simple complex roots of g.

(i)(b): From (4), it follows that z_- or z_+ is a root of g. In view of Fact 2.1 and $g'(z_-) = g'(z_+)$, it follows that one of z_-, z_+ is at least a double root, but not both; moreover, it cannot be a triple root because 0 is the only root of g'' and $z_- < 0 < z_+$. Hence exactly one of z_-, z_+ is a double root. To verify the remaining parts, we first define

$$r_1 := \frac{3q}{p} \quad \text{and} \quad r_2 := \frac{-3q}{2p}.$$

Because $\Delta = 0$, it follows that $4p^3 + 27q^2 = 0$. Hence

$$g(r_1) = r_1^3 + pr_1 + q = \frac{27q^3}{p^3} + \frac{3pq}{p} + q = \frac{27q^3}{p^3} + 4q = \frac{q}{p^3}\big(27q^2 + 4p^3\big) = 0$$

and

$$g(r_2) = r_2^3 + pr_2 + q = \frac{-27q^3}{8p^3} + \frac{-3pq}{2p} + q = \frac{-27q^3}{8p^3} - \frac{q}{2} = \frac{-q}{8p^3}\big(27q^2 + 4p^3\big) = 0.$$

The assumption that $\Delta = 0$ readily yields

$$p = \frac{-3^{1/3}q^{2/3}}{2^{2/3}} \quad \text{and} \quad |q| = \frac{2(-p)^{3/2}}{3^{3/2}}.$$

Hence $\qquad r_1 = 3qp^{-1} = 3q(-1)3^{-1/3}q^{-2/3}2^{2/3} = 2^{2/3}(-q)^{1/3}$

and $\qquad r_2 = -3q2^{-1}p^{-1} = -3q2^{-1}(-1)3^{-1/3}q^{-2/3}2^{2/3} = -2^{-1/3}(-q)^{1/3}$

as claimed. If $q > 0$, then

$$r_1 = \frac{3q}{p} = \frac{3 \cdot 2(-p)^{3/2}}{3^{3/2}p} = -2(-p/3)^{1/2} = 2z_-$$

and $$r_2 = \frac{-3q}{2p} = -\frac{1}{2}\,\frac{3q}{p} = -\frac{1}{2}r_1 = -\frac{1}{2}2z_- = z_+.$$

Similarly, if $q < 0$, then $r_1 = 2z_+$ and $r_2 = z_-$.

No matter the sign of q, we have $r_2 \in \{z_-, z_+\}$ and thus $g'(r_2) = 0$, i.e., r_2 is the double root.

(i)(c): In view of (4), $g(z_-)$ and $g(z_+)$ have opposite signs. Because g is strictly decreasing on $[z_-, z_+]$, it follows that $g(z_-) > 0 > g(z_+)$. Hence there is at least on real root r_0 in $]z_-, z_+[$. On the other hand, g is strictly increasing on $]-\infty, z_-]$ and on $[z_+, +\infty[$ which yields further roots r_- and r_+ as announced. Having now three real roots, they must all be simple.

Next, note that

$$\begin{aligned}\Delta < 0 &\Leftrightarrow 0 \le (q/2)^2 < -(p/3)^3 = (-p/3)^3 \Leftrightarrow 0 \le (q/2)^2/(-p/3)^3 < 1 \\ &\Leftrightarrow 0 \le (|q|/2)/(-p/3)^{3/2} < 1 \Leftrightarrow -1 < (-q/2)/(-p/3)^{3/2} < 1.\end{aligned}$$

It follows that $$\theta = \arccos\frac{-q/2}{(-p/3)^{3/2}} \in\,]0, \pi[\tag{79}$$

as claimed. For convenience, we set, for $k \in \{0, 1, 2\}$,

$$\theta_k := \frac{\theta + 2k\pi}{3}; \quad \text{hence,}\ z_k = 2(-p/3)^{1/2}\cos(\theta_k). \tag{80}$$

Recall that $0 < \theta < \pi$, which allows us to draw three conclusions:

$$0 < \theta_0 = \theta/3 < \pi/3 \Rightarrow 1 > \cos(\theta_0) = \cos(\theta/3) > 1/2; \tag{81a}$$

$$2\pi/3 < \theta_1 = (\theta+2\pi)/3 < \pi \Rightarrow -1/2 > \cos(\theta_1) = \cos((\theta+2\pi)/3) > -1; \tag{81b}$$

$$4\pi/3 < \theta_2 = (\theta+4\pi)/3 < 5\pi/3 \Rightarrow -1/2 < \cos(\theta_2) = \cos((\theta+2\pi)/3) < 1/2. \tag{81c}$$

Hence $\cos(\theta_1) < \cos(\theta_2) < \cos(\theta_0)$ and thus

$$z_1 < z_2 < z_0. \tag{82}$$

All we need to do is to verify that each z_k is actually a root of g. To this end, observe first that the triple-angle formula for the cosine (see, e.g., [1, Formula 4.3.28 on p. 72]) yields

$$\cos^3(\theta_k) = \frac{3\cos(\theta_k) + \cos(3\theta_k)}{4} = \frac{3\cos(\theta_k) + \cos(\theta + 2k\pi)}{4} \tag{83a}$$

$$= \frac{3\cos(\theta_k) + \cos(\theta)}{4}. \tag{83b}$$

Then

$$\begin{aligned} g(z_k) &= z_k^3 + pz_k + q \\ &= 8(-p/3)^{3/2}\cos^3(\theta_k) + p2(-p/3)^{1/2}\cos(\theta_k) + q \\ &= 2(-p/3)^{3/2}\big(3\cos(\theta_k) + \cos(\theta)\big) + 2(-p/3)^{1/2}p\cos(\theta_k) + q \qquad \text{(using (83))} \\ &= 2(-p/3)^{1/2}\cos(\theta_k)\big(3(-p/3) + p\big) + 2(-p/3)^{3/2}\cos(\theta) + q \\ &= 2(-p/3)^{3/2}\cos(\theta) + q \\ &= 2(-p/3)^{3/2}\frac{-q/2}{(-p/3)^{3/2}} + q \qquad \text{(using (79))} \\ &= 0, \end{aligned}$$

and this completes the proof for this case.

(ii): If $q = 0$, then $g(z) = z^3$ so $z = 0$ is the only root of g and it is of multiplicity 3. Thus we assume that $q \neq 0$. Then

$$g(z) = 0 \;\Leftrightarrow\; z^3 + q = 0 \;\Leftrightarrow\; z^3 = -q \;\Rightarrow\; z = (-q)^{1/3} \neq 0.$$

Because g is strictly increasing on $\mathbb{R}$, $r := (-q)^{1/3}$ is the only real root of g. Because g' has only one real root, namely 0, it follows that $g'(r) \neq 0$ and so r is a simple root. Denoting again by ω the primitive 3rd root of unity (see (78)), it is clear that the remaining complex (simple) roots are ωr and $\overline{\omega} r$ as claimed.

(iii): Note that $\Delta \geq (p/3)^3 > 0$ because $p > 0$. The fact that r is a root is shown exactly as in (i)(a). It is simple because g' has no real roots, and r is unique because g is strictly increasing. The complex roots are derived exactly as in (i)(a).

This concludes the proof.

B. Proofs for Section 5

B.1. Proof of Proposition 5.1

Note that $h'(x) = 4\alpha x^3 + 3\beta x^2 + 2\gamma x + \delta$ and, also completing the square,

$$h''(x) = 12\alpha x^2 + 6\beta x + 2\gamma = \frac{3}{4}\alpha\Big(4x + \frac{\beta}{\alpha}\Big)^2 + 2\gamma - \frac{3\beta^2}{4\alpha}. \tag{84}$$

Hence $h'' \geq 0 \Leftrightarrow [\alpha > 0 \text{ and } 2\gamma \geq 3\beta^2/(4\alpha)] \Leftrightarrow$ (38). (For further information on deciding nonnegativity of polynomials, see [6, Section 3.1.3].) This concludes the proof.

B.2. Proof of Proposition 5.2

Because h is supercoercive, it follows from [3, Proposition 14.15] that $\operatorname{dom} h^* = \mathbb{R}$. Combining with the differentiability of h, it follows that $y \in \operatorname{int}\operatorname{dom} h^* \subseteq \operatorname{dom}\partial h^* = \operatorname{ran}\partial h = \operatorname{ran} h'$. However, if $h'(x) = y$, then $h^*(y) = xy - h(x)$ and we have found the conjugate. It remains to solve $h'(x) = y$, i.e.,

$$4\alpha x^3 + 3\beta x^2 + 2\gamma x + \delta - y = 0. \tag{85}$$

So we set

$$f(x) := ax^3 + bx^2 + cx + d, \quad \text{where } a := 4\alpha,\ \ b := 3\beta,\ \ c := 2\gamma,\ \ d := \delta - y.$$

To solve (85), i.e., $f(x) = 0$, we first note that

$$p = \frac{3ac - b^2}{3a^2} = \frac{3(4\alpha)(2\gamma) - (3\beta)^2}{3(4\alpha)^2} = \frac{8\alpha\gamma - 3\beta^2}{16\alpha^2} \geq 0,$$

where the inequality follows from (39). Next,

$$\begin{aligned} q &= \frac{3^3a^2d + 2b^3 - 3^2abc}{(3a)^3} = \frac{3^34^2\alpha^2(\delta - y) + 2(3^3\beta^3) - 3^2(4\alpha)(3\beta)(2\gamma)}{3^34^3\alpha^3} \\ &= \frac{8\alpha^2(\delta - y) + \beta^3 - 4\alpha\beta\gamma}{32\alpha^3} \end{aligned}$$

and

$$\Delta = (p/3)^3 + (q/2)^2 \geq 0,$$

where the inequality follows because $p \geq 0$. Then $-b/(3a) = -\beta/(4\alpha)$ and now Corollary 4.5(i) yields the unique solution of $f(x) = 0$ as (41). This concludes the proof.

B.3. Proof of Proposition 5.3

Because h is differentiable and full domain, it follows that $\operatorname{Prox}_h(y)$ is the *unique* solution x of the equation $h'(x) + x - y = 0$. The proof thus proceeds analogously to that of Proposition 5.2 – the only difference is we must solve

$$f(x) := ax^3 + bx^2 + cx + d, \quad \text{where } a := 4\alpha,\ \ b := 3\beta,\ \ c := 2\gamma + 1,\ \ d := \delta - y.$$

(The only difference is that $c = 2\gamma + 1$ rather than 2γ due to the additional term "$+x$".) Thus we know *a priori* that the resulting cubic must have a unique real solution. We now have

$$\begin{aligned} 0 &< \frac{1}{4\alpha} \leq \frac{1}{4\alpha} + \frac{8\alpha\gamma - 3\beta^2}{16\alpha^2} = \frac{4\alpha(1 + 2\gamma) - 3\beta^2}{16\alpha^2} = p = \frac{12\alpha(1 + 2\gamma) - 9\beta^2}{48\alpha^2} \\ &= \frac{3ac - b^2}{3a^2}, \end{aligned}$$

which is our usual p from discussing roots of the cubic f. Similarly, the q defined here is the same as the usual q for $f(x)$ (see (21)). Finally, the formula for $x = \operatorname{Prox}_h(y)$ now follows from Corollary 4.5(i). This concludes the proof.

B.4. Proof of Example 5.4

Note that h fits the pattern of (37) with $\alpha = \beta = \gamma = \delta = \varepsilon = 1$. The characterization of convexity presented in Proposition 5.1 turns into $1 > 0$ and $8 \geq 3$ which are both obviously true. Hence h is convex.

To compute the Fenchel conjugate, we apply Proposition 5.2 and get $p = 5/16$, $q = (5 - 8y)/32 = -(y - 5/8)/4$, and $\Delta = 5^3/16^3 + (y - 5/8)^2/8^2$. It follows $-q/2 = (y - 5/8)/8$. Hence (41) turns into

$$\begin{aligned} x_y &= -\frac{1}{4} + \sqrt[3]{\frac{y-5/8}{8} + \sqrt{\frac{(y-5/8)^2}{8^2} + \frac{5^3}{16^3}}} + \sqrt[3]{\frac{y-5/8}{8} - \sqrt{\frac{(y-5/8)^2}{8^2} + \frac{5^3}{16^3}}} \\ &= -\frac{1}{4} + \sqrt[3]{\frac{y-5/8}{8} + \sqrt{\frac{(y-5/8)^2}{8^2} + \frac{5^3}{8^2\cdot 4^3}}} + \sqrt[3]{\frac{y-5/8}{8} - \sqrt{\frac{(y-5/8)^2}{8^2} + \frac{5^3}{8^2\cdot 4^3}}}, \end{aligned}$$

which simplifies to (45a).

To compute $\mathrm{Prox}_h(y)$, we utilize Proposition 5.3. Obtaining fresh values for p, q, Δ, we have this time

$$p = 9/16,\ q = (3 - 8y)/32,\ \Delta = ((8y-3)^2 + 27)/4096 = ((8y-3)^2 + 3^3)/64^2.$$

Hence $-q/2 = (8y - 3)/64$ and $\sqrt{\Delta} = \sqrt{(8y-3)^2 + 3^3}/64$. It follows that

$$\begin{aligned} \sqrt[3]{\frac{-q}{2} \pm \sqrt{\Delta}} &= \sqrt[3]{\frac{8y-3}{64} \pm \frac{\sqrt{(8y-3)^2+3^3}}{64}} = \frac{1}{4}\sqrt[3]{8y - 3 \pm \sqrt{(8y-3)^2 + 3^3}} \\ &= \frac{1}{2}\sqrt[3]{y - \frac{3}{8} \pm \sqrt{(y - \frac{3}{8})^2 + (\frac{3}{4})^3}}. \end{aligned}$$

This, $-\beta/(4\alpha) = -1/4$, and (42) now yields (46). This concludes the proof.

B.5. Proof of Example 5.5

Note that h fits the pattern of (37) with $\beta = \gamma = \delta = \varepsilon = 0$. The characterization of convexity presented in Proposition 5.1 turns into $\alpha > 0$ and $0 \geq 0$ which are both obviously true. Hence h is convex.

We start by computing the Fenchel conjugate of h using Proposition 5.2. We have $p = 0$, $q = -y/(4\alpha)$, $\Delta = (y/(8\alpha))^2$, and $-\beta/(4\alpha) = 0$. Hence $-q/2 = y/(8\alpha)$ and $\sqrt{\Delta} = |y|/(8\alpha)$ which imply

$$\sqrt[3]{(-q/2) \pm \sqrt{\Delta}} = \sqrt[3]{y/(8\alpha) \pm |y|/(8\alpha)} = \sqrt[3]{\max\{0, y/(4\alpha)\}} \text{ or } \sqrt[3]{\min\{0, y/(4\alpha)\}}.$$

Using (41), we get

$$x_y = \sqrt[3]{\max\{0, y/(4\alpha)\}} + \sqrt[3]{\min\{0, y/(4\alpha)\}} = \sqrt[3]{y/(4\alpha)}.$$

Using Proposition 5.2, we obtain

$$\begin{aligned} h^*(y) &= yx_y - h(x_y) = yy^{1/3}/(4\alpha)^{1/3} - \alpha y^{4/3}/(4\alpha)^{4/3} \\ &= \frac{|y|^{4/3}}{4^{1/3}\alpha^{1/3}} - \frac{|y|^{4/3}}{4^{4/3}\alpha^{1/3}} = \frac{|y|^{4/3}}{4^{1/3}\alpha^{1/3}}\big(1 - \tfrac{1}{4}\big) = \frac{3|y|^{4/3}}{4(4\alpha)^{1/3}} \end{aligned}$$

as claimed.

To compute $\operatorname{Prox}_h(y)$, we utilize Proposition 5.3. Obtain fresh values of p, q, Δ, we have this time $p = 1/(4\alpha) > 0$ and $q = -y/(4\alpha)$ (see (43)). Hence

$$\Delta = (p/3)^3 + (q/2)^2 = (1 + 27\alpha y^2)/(1728\alpha^3) \text{ and } \sqrt{\Delta} = \sqrt{1 + 27\alpha y^2}/(8(3\alpha)^{3/2}).$$

Now $-\beta/(4\alpha) = 0$ and $-q/2 = y/(8\alpha)$, so (42) yields

$$\begin{aligned}\operatorname{Prox}_h(y) &= \sqrt[3]{\frac{y}{8\alpha} + \frac{\sqrt{1+27\alpha y^2}}{8(3\alpha)^{3/2}}} + \sqrt[3]{\frac{y}{8\alpha} - \frac{\sqrt{1+27\alpha y^2}}{8(3\alpha)^{3/2}}} \\ &= \frac{1}{2}\sqrt[3]{\frac{y}{\alpha} + \sqrt{\frac{1+27\alpha y^2}{27\alpha^3}}} + \frac{1}{2}\sqrt[3]{\frac{y}{\alpha} - \sqrt{\frac{1+27\alpha y^2}{27\alpha^3}}}\end{aligned}$$

and this concludes the proof.

C. Proof of Proposition 6.1

Because we have $\operatorname{dom} h = \mathbb{R}_{++}$, we must find the *positive* solution of the equation $h'(x) + x - y = 0$. Since $h'(x) = -\alpha x^{-2}$, we are looking for the (necessarily unique) positive solution of $x^2(h'(x) + x - y) = 0$, i.e., of

$$x^3 - yx^2 - \alpha = 0.$$

This fits the pattern of (18) in Section 4, with parameters $a = 1$, $b = -y$, $c = 0$, and $d = -\alpha$. As in (21), we set

$$p := \frac{3ac - b^2}{3a^2} = -\frac{y^2}{3}\begin{cases} < 0, & \text{if } y \neq 0; \\ = 0, & \text{if } y = 0\end{cases} \tag{86}$$

and
$$q := \frac{27a^2 d + 2b^3 - 9abc}{27a^3} = -\alpha - 2(y/3)^3.$$

Next,
$$\begin{aligned}\Delta &= (p/3)^3 + (q/2)^2 = -y^6/9^3 + (\alpha + 2(y/3)^3)^2/4 \\ &= -(y/3)^6 + \alpha^2/4 + \alpha(y/3)^3 + (y/3)^6 \\ &= \alpha\big(\alpha/4 + (y/3)^3\big).\end{aligned}$$

Hence
$$\Delta\begin{cases} < 0 \Leftrightarrow y < y_0; \\ = 0 \Leftrightarrow y = y_0; \\ > 0 \Leftrightarrow y > y_0,\end{cases} \qquad \text{where } y_0 := -\frac{3}{\sqrt[3]{4}}\sqrt[3]{\alpha} \approx -1.88988\sqrt[3]{\alpha}. \tag{87}$$

Now set
$$x_0 := -\frac{b}{3a} = \frac{y}{3}. \tag{88}$$

Note that
$$-q/2 = (y/3)^3 + \alpha/2. \tag{89}$$

We now discuss the three possibilities from Corollary 4.5 – these will correspond to the three items of the result!

Case 1: $b^2 = 3ac$ or $\Delta > 0$; equivalently, $y = 0$ or $y > y_0$; equivalently, $y_0 < y$. Then Corollary 4.5(i) yields, as claimed,

$$\begin{aligned}\operatorname{Prox}_h(y) &= x_0 + \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}} \\ &= \frac{y}{3} + \sqrt[3]{\frac{\alpha}{2} + \Big(\frac{y}{3}\Big)^3 + \sqrt{\alpha\Big(\frac{\alpha}{4} + \Big(\frac{y}{3}\Big)^3\Big)}} + \sqrt[3]{\frac{\alpha}{2} + \Big(\frac{y}{3}\Big)^3 - \sqrt{\alpha\Big(\frac{\alpha}{4} + \Big(\frac{y}{3}\Big)^3\Big)}}.\end{aligned}$$

Case 2: $\Delta = 0$; equivalently, $y = y_0$.

Then Corollary 4.5(ii) yields two distinct real roots. We can take a short cut here, though: By exploiting the continuity of Prox_h at y_0 via $\mathrm{Prox}_h(y_0) = \lim_{y\to y_0^+} \mathrm{Prox}_h(y)$, we get

$$\mathrm{Prox}_h(y_0) = \frac{y_0}{3} + 2\sqrt[3]{\frac{\alpha}{2} + \left(\frac{y_0}{3}\right)^3} = \sqrt[3]{\alpha}/\sqrt[3]{4} \approx 0.62996\sqrt[3]{\alpha}.$$

Case 3: $\Delta < 0$; equivalently, $y < y_0$.

By uniqueness of $\mathrm{Prox}_h(y)$, the desired root must be the *largest* (and the only positive) real root offered in this case (see Corollary 4.5(iii)):

$$\mathrm{Prox}_h(y) = x_0 + 2(-p/3)^{1/2}\cos\left(\frac{\theta}{3}\right), \quad \text{where } \theta := \arccos\frac{-q/2}{(-p/3)^{3/2}}, \tag{90}$$

By (86), $-p/3 = y^2/9$; thus, using $y < y_0 < 0$, we obtain $(-p/3)^{1/2} = -y/3$, $(-p/3)^{3/2} = -(y/3)^3$, and (89) yields $-(q/2)/(-p/3)^{3/2} = -((y/3)^3 + \alpha/2)/(y/3)^3$. This and (88) results in

$$\mathrm{Prox}_h(y) = \frac{y}{3} - 2\frac{y}{3}\cos\left(\frac{\theta}{3}\right), \quad \text{where } \theta := \arccos\left(-\frac{(y/3)^3 + \alpha/2}{(y/3)^3}\right). \tag{91}$$

The proof is complete.

D. Proof of Theorem 7.1

For $x \geq 0$, we have $xh = x\alpha\|\cdot\|^2$, $x\nabla h = 2\alpha x\,\mathrm{Id}$, $\mathrm{Id} + x\nabla h = (1+2\alpha x)\,\mathrm{Id}$ and therefore $\mathrm{Prox}_{xh} = (1+2\alpha x)^{-1}\,\mathrm{Id}$. In view of [5, Theorem 6.36], we must first find a positive root x of $\varphi(x) := h(\mathrm{Prox}_{xh}(\mathbf{y})) - x - \eta = \alpha\|\mathbf{y}\|^2/(1+2\alpha x)^2 - x - \eta = 0$. Note that $\varphi(0) > 0$, that φ is strictly decreasing on $\mathbb{R}_+$, and that $\varphi(x) \to -\infty$ as $x \to +\infty$. Hence φ has *exactly one* positive root. Multiplying by $(1+2\alpha x)^2 > 0$, where $x > 0$, results in the cubic $\alpha\nu^2 - (x+\eta)(1+2\alpha x)^2 = 0$, which must have *exactly one* positive root. Re-arranging, we are led to

$$f(x) := 4\alpha^2x^3 + 4\alpha(\alpha\eta+1)x^2 + (4\alpha\eta+1)x + \eta - \alpha\nu^2 = 0, \tag{92}$$

a cubic which we know has exactly one positive root. As in Section 4, we set

$$a := 4\alpha^2, \ \ b := 4\alpha(\alpha\eta+1), \ \ c := 4\alpha\eta+1, \ \ d := \eta - \alpha\nu^2 < 0, \tag{93}$$

$$p := \frac{3ac-b^2}{3a^2} = -\frac{(2\alpha\eta-1)^2}{12\alpha^2} \leq 0, \tag{94}$$

and

$$q := \frac{27a^2d + 2b^3 - 9abc}{27a^3} = \frac{8\alpha^3\eta^3 - 12\alpha^2\eta^2 + 6\alpha\eta - 27\alpha^2\nu^2 - 1}{108\alpha^3}. \tag{95}$$

We then have, as claimed,

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{\big(8\alpha^3\eta^3 - 12\alpha^2\eta^2 + 6\alpha\eta - 27\alpha^2\nu^2 - 1\big)^2 - (2\alpha\eta-1)^6}{(6\alpha)^6} \tag{96a}$$

$$= -\frac{\nu^2}{1728\alpha^4}\big(16\alpha^3\eta^3 - 24\alpha^2\eta^2 + 12\alpha\eta - 27\alpha^2\nu^2 - 2\big) \tag{96b}$$

$$= \frac{\nu^2}{1728\alpha^4}\big(27\alpha^2\nu^2 - 2(2\alpha\eta-1)^3\big),. \tag{96c}$$

Utilizing Corollary 4.5, we have

$$x = -\frac{\alpha\eta+1}{3\alpha} + \sqrt[3]{-q/2+\sqrt{\Delta}} + \sqrt[3]{-q/2-\sqrt{\Delta}}, \quad \text{if } \Delta \geq 0 \tag{97a}$$

and

$$x = -\frac{\alpha\eta+1}{3\alpha} + \frac{|2\alpha\eta-1|}{3\alpha}\cos\Big(\frac{1}{3}\arccos\frac{-q/2}{(-p/3)^{3/2}}\Big) \quad \text{if } \Delta < 0. \tag{97b}$$

(Because we know there is *exactly one* positive root, it is clear that we must pick r_0 in Corollary 4.5(iii) when $\Delta < 0$.) Finally, [5, Theorem 6.36] yields

$$P_E(\mathbf{y},\eta) = \big(\operatorname{Prox}_{xh}(\mathbf{y}), \eta + x\big) = \Big(\frac{\mathbf{y}}{1+2\alpha x}, \eta + x\Big) \tag{98}$$

as claimed.

E. Proofs for Section 8

E.1. Proof of Theorem 8.1

By [4, Theorem 4.1(ii)(a)], there exists a *unique* $x \in \,]-1,1[$ such that

$$\frac{2\zeta^2}{(1-x)^2} + \frac{2\alpha^2 x}{\beta^2} + 2\alpha\gamma = 0; \tag{99}$$

multiplying by $\beta^2(1-x)^2/2 > 0$ yields the cubic

$$f(x) := ax^3 + bx^2 + cx + d = 0 \tag{100}$$

where

$$a := \alpha^2 > 0, \;\; b := \alpha\beta^2\gamma - 2\alpha^2, \;\; c := \alpha^2 - 2\alpha\beta^2\gamma, \;\; d := \alpha\beta^2\gamma + \beta^2\zeta^2. \tag{101}$$

Our strategy is to systematically discuss all cases of Theorem 4.1 and then combine cases as much as possible. As usual, we set

$$x_0 := -\frac{b}{3a} = \frac{2\alpha - \beta^2\gamma}{3\alpha} = \frac{2\alpha^2 - \alpha\beta^2\gamma}{3\alpha^2} \quad \text{and } p := \frac{3ac-b^2}{3a^2} = -\frac{(\alpha+\beta^2\gamma)^2}{3\alpha^2} \leq 0, \tag{102}$$

and we note that the definition of p is consistent with the one given in (61). We have the characterization

$$p = 0 \;\Leftrightarrow\; \alpha + \beta^2\gamma = 0 \;\Leftrightarrow\; \gamma = -\alpha/\beta^2. \tag{103}$$

Again as usual, we set

$$q := \frac{27a^2d + 2b^2 - 9abc}{27a^3} = \frac{2(\alpha+\beta^2\gamma)^3}{27\alpha^3} + \frac{\beta^2\zeta^2}{\alpha^2}, \tag{104}$$

which matches (61), and of course

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{\beta^2\zeta^2}{\alpha^2}\Big(\frac{\beta^2\zeta^2}{4\alpha^2} + \frac{(\alpha+\beta^2\gamma)^3}{27\alpha^3}\Big), \tag{105}$$

which matches (62).

We now systematically discuss the case of Theorem 4.1.

Case 1: $p < 0$, i.e., $\alpha + \beta^2\gamma \neq 0$ by (61).

Case 1(a): $p < 0$ and $\Delta > 0$.
Then Theorem 4.1(i)(a) and the definition x_0 in (102) yield (63).

Case 1(b): $p < 0$ and $\Delta = 0$.
By Theorem 4.1(i)(b), there are two roots, $x_0 + 3q/p$ and $x_0 - 3q/(2p)$, one of which lies in $]-1, 1[$. Now

$$\begin{aligned} x_0 - \frac{3q}{2p} - 1 &= \frac{2\alpha - \beta^2\gamma}{3\alpha} - \frac{3}{2}\frac{2(\alpha+\beta^2\gamma)^3 + 27\alpha\beta^2\zeta^2}{27\alpha^3}\frac{-3\alpha^2}{(\alpha+\beta^2\gamma)^2} - 1 \\ &= \frac{2\alpha - \beta^2\gamma}{3\alpha} + \frac{(\alpha+\beta^2\gamma)^3 + 27\alpha\beta^2\zeta^2/2}{3\alpha(\alpha+\beta^2\gamma)^2} - 1 \\ &= \frac{\big((2\alpha - \beta^2\gamma) + (\alpha+\beta^2\gamma) - (3\alpha)\big)}{3\alpha(\alpha+\beta^2\gamma)^2}(\alpha+\beta^2\gamma)^2 + \frac{27\alpha\beta^2\zeta^2/2}{3\alpha(\alpha+\beta^2\gamma)^2} \\ &= \frac{9\beta^2\zeta^2}{2(\alpha+\beta^2\gamma)^2} \geq 0; \end{aligned}$$

hence the root $x_0 - 3q/(2p)$ lies in $[1, +\infty[$ and therefore our desired root is the remaining one, namely $x_0 + 3q/p$, which also allows us to use the representation (63).

Case 1(c): $p < 0$ and $\Delta < 0$.
According to Theorem 4.1(i)(c), we have three distinct real roots, but there is information about their location. We must locate the root in $]-1, 1[$. First, $b^2 - 3ac = (\alpha^2 + \alpha\beta^2\gamma)^2$ which yields $\sqrt{b^2 - 3ac} = |\alpha^2 + \alpha\beta^2\gamma|$. This and the definition of b yields

$$\begin{aligned} x_\pm := \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} &= \frac{2\alpha^2 - \alpha\beta^2\gamma \pm |\alpha^2 + \alpha\beta^2\gamma|}{3\alpha^2} \\ &= \frac{1}{3\alpha^2}\frac{(3\alpha^2) + (\alpha^2 - 2\alpha\beta^2\gamma) \pm \big|(3\alpha^2) - (\alpha^2 - 2\alpha\beta^2\gamma)\big|}{2}. \end{aligned}$$

Hence
$$x_- = \frac{\min\{3\alpha^2, \alpha^2 - 2\alpha\beta^2\gamma\}}{3\alpha^2} < \frac{\max\{3\alpha^2, \alpha^2 - 2\alpha\beta^2\gamma\}}{3\alpha^2} = x_+. \tag{106}$$

We now bifurcate one last time.

Case 1(c)(+): $p < 0$, $\Delta < 0$, and $\alpha^2 + \alpha\beta^2\gamma > 0$.
Then $3\alpha^2 > \alpha^2 - 2\alpha\beta^2\gamma$ and therefore $x_+ = 1$. It follows that our desired root x is the "middle root" corresponding to $k = 2$ in Theorem 4.1(i)(c):

$$\begin{aligned} x &= x_0 + 2(-p/3)^{1/2}\cos\left(\frac{1}{3}\left(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\right)\right) \\ &= \frac{2\alpha - \beta^2\gamma}{3\alpha} + \frac{2|\alpha+\beta^2\gamma|}{3|\alpha|}\cos\left(\frac{1}{3}\left(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\right)\right) \\ &= \frac{2\alpha - \beta^2\gamma}{3\alpha} + \frac{2(\alpha+\beta^2\gamma)}{3\alpha}\cos\left(\frac{1}{3}\left(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\right)\right), \end{aligned}$$

where in the last line we used the assumption to deduce that

$$|\alpha+\beta^2\gamma|/|\alpha| = |\alpha^2 + \alpha\beta^2\gamma|/\alpha^2 = (\alpha^2 + \alpha\beta^2\gamma)/\alpha^2 = (\alpha+\beta^2\gamma)/\alpha.$$

Case 1(c)(−): $p < 0$, $\Delta < 0$, and $\alpha^2 + \alpha\beta^2\gamma \le 0$.
Then $3\alpha^2 \le \alpha^2 - 2\alpha\beta^2\gamma$ and therefore $x_- = 1$. It follows that our desired root is the "smallest root" corresponding to $k = 1$ in Theorem 4.1(i)(c):

$$\begin{aligned} x &= x_0 + 2(-p/3)^{1/2}\cos\Big(\frac{1}{3}\Big(2\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= \frac{2\alpha-\beta^2\gamma}{3\alpha} + \frac{2|\alpha+\beta^2\gamma|}{3|\alpha|}\cos\Big(\frac{1}{3}\Big(2\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= \frac{2\alpha-\beta^2\gamma}{3\alpha} - \frac{2(\alpha+\beta^2\gamma)}{3\alpha}\cos\Big(\frac{1}{3}\Big(2\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big), \end{aligned}$$

where in the last line we used the assumption to deduce that

$$|\alpha+\beta^2\gamma|/|\alpha| = |\alpha^2+\alpha\beta^2\gamma|/\alpha^2 = -(\alpha^2+\alpha\beta^2\gamma)/\alpha^2 = -(\alpha+\beta^2\gamma)/\alpha.$$

Note that the last two cases can be combined to obtain (64).

Case 2: $p = 0$, i.e., $\alpha + \beta^2\gamma = 0$ by (61).
Then $\Delta = (q/2)^2 \ge 0$; hence, $\sqrt{\Delta} = |q|/2$ and thus $\{-q/2 \pm \sqrt{\Delta}\} = \{-q, 0\}$. By Theorem 4.1(ii), the only real root is

$$x_0 + (-q)^{1/3} = x_0 + (-q/2 + \sqrt{\Delta})^{1/3} + (-q/2 - \sqrt{\Delta})^{1/3}$$

which is the same as (63) using (102).

Case 3: $p > 0$. In view of (61), this case never occurs and we are done.

E.2. Proof of Theorem 8.2

By [4, Theorem 4.1(iii)(a)], there exists a *unique* $x \in \,]-1, 1[$ such that

$$\frac{2\zeta^2}{(1+x)^2} - \frac{2\alpha^2 x}{\beta^2} - 2\alpha\gamma = 0; \tag{107}$$

multiplying by $-\beta^2(1+x)^2/2 < 0$ yields the cubic

$$f(x) := ax^3 + bx^2 + cx + d = 0, \quad \text{where} \tag{108}$$

$$a := \alpha^2 > 0, \;\; b := \alpha\beta^2\gamma + 2\alpha^2, \;\; c := \alpha^2 + 2\alpha\beta^2\gamma, \;\; d := \alpha\beta^2\gamma - \beta^2\zeta^2. \tag{109}$$

Our strategy is to systematically discuss all cases of Theorem 4.1 and then combine cases as much as possible. As usual, we set

$$x_0 := -\frac{b}{3a} = -\frac{2\alpha+\beta^2\gamma}{3\alpha} = -\frac{2\alpha^2+\alpha\beta^2\gamma}{3\alpha^2} \;\text{ and }\; p := \frac{3ac-b^2}{3a^2} = -\frac{(\beta^2\gamma-\alpha)^2}{3\alpha^2} \le 0, \tag{110}$$

and we note that the definition of p is consistent with the one given in (68). We have the characterization

$$p = 0 \;\Leftrightarrow\; \beta^2\gamma - \alpha = 0 \;\Leftrightarrow\; \gamma = \alpha/\beta^2. \tag{111}$$

Again as usual, we set

$$q := \frac{27a^2d + 2b^2 - 9abc}{27a^3} = \frac{2(\beta^2\gamma-\alpha)^3}{27\alpha^3} - \frac{\beta^2\zeta^2}{\alpha^2}, \tag{112}$$

which matches (68), and of course

$$\Delta := (p/3)^3 + (q/2)^2 = \frac{\beta^2\zeta^2}{\alpha^2}\Big(\frac{\beta^2\zeta^2}{4\alpha^2} - \frac{(\beta^2\gamma-\alpha)^3}{27\alpha^3}\Big), \tag{113}$$

which matches (69).

We now systematically discuss the case of Theorem 4.1.

Case 1: $p < 0$, i.e., $\beta^2\gamma - \alpha \neq 0$ by (68).

Case 1(a): $p < 0$ and $\Delta > 0$.
Then Theorem 4.1(i)(a) and the definition of x_0 in (110) yield (70).

Case 1(b): $p < 0$ and $\Delta = 0$.
By Theorem 4.1(i)(b), there are two roots, $x_0 + 3q/p$ and $x_0 - 3q/(2p)$, one of which lies in $]-1, 1[$. Now

$$\begin{aligned} x_0 - \frac{3q}{2p} + 1 &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} - \frac{3}{2}\frac{2(\beta^2\gamma-\alpha)^3 - 27\alpha\beta^2\zeta^2}{27\alpha^3}\frac{-3\alpha^2}{(\beta^2\gamma-\alpha)^2} + 1 \\ &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} + \frac{(\beta^2\gamma-\alpha)^3 - 27\alpha\beta^2\zeta^2/2}{3\alpha(\beta^2\gamma-\alpha)^2} + 1 \\ &= \frac{\big((-2\alpha-\beta^2\gamma) + (\beta^2\gamma-\alpha) + (3\alpha)\big)}{3\alpha(\beta^2\gamma-\alpha)^2}(\beta^2\gamma-\alpha)^2 - \frac{27\alpha\beta^2\zeta^2/2}{3\alpha(\beta^2\gamma-\alpha)^2} \\ &= -\frac{9\beta^2\zeta^2}{2(\beta^2\gamma-\alpha)^2} \leq 0; \end{aligned}$$

hence the root $x_0 - 3q/(2p)$ lies in $]-\infty, -1]$ and therefore our desired root is the remaining one, namely $x_0 + 3q/p$, which also allows us to use the representation (70).

Case 1(c): $p < 0$ and $\Delta < 0$.
According to Theorem 4.1(i)(c), we have three distinct real roots, but there is information about their location. We must locate the root in $]-1, 1[$. First, $b^2 - 3ac = (\alpha^2 - \alpha\beta^2\gamma)^2$ which yields $\sqrt{b^2 - 3ac} = |\alpha^2 - \alpha\beta^2\gamma|$. This and the definition of b yields

$$\begin{aligned} x_\pm : &= \frac{-b \pm \sqrt{b^2-3ac}}{3a} = \frac{-2\alpha^2 - \alpha\beta^2\gamma \pm |\alpha^2 - \alpha\beta^2\gamma|}{3\alpha^2} \\ &= \frac{1}{3\alpha^2}\frac{(-3\alpha^2) + (-\alpha^2 - 2\alpha\beta^2\gamma) \pm \big|(-3\alpha^2) - (-\alpha^2 - 2\alpha\beta^2\gamma)\big|}{2}. \end{aligned}$$

Hence $$x_- = \frac{\min\{-3\alpha^2, -\alpha^2 - 2\alpha\beta^2\gamma\}}{3\alpha^2} < \frac{\max\{-3\alpha^2, -\alpha^2 - 2\alpha\beta^2\gamma\}}{3\alpha^2} = x_+. \tag{114}$$

We now bifurcate one last time.

Case 1(c)(+): $p < 0$, $\Delta < 0$, and $\alpha^2 - \alpha\beta^2\gamma > 0$.
Then $-3\alpha^2 < -\alpha^2 - 2\alpha\beta^2\gamma$ and therefore $x_- = -1$. It follows that our desired root x is the "middle root" corresponding to $k = 2$ in Theorem 4.1(i)(c):

$$\begin{aligned} x &= x_0 + 2(-p/3)^{1/2} \cos\Big(\frac{1}{3}\Big(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} + \frac{2|\beta^2\gamma-\alpha|}{3|\alpha|} \cos\Big(\frac{1}{3}\Big(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} + \frac{2(\alpha-\beta^2\gamma)}{3\alpha} \cos\Big(\frac{1}{3}\Big(4\pi + \arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big), \end{aligned}$$

where in the last line we used the assumption to deduce that

$$|\beta^2\gamma-\alpha|/|\alpha| = |\alpha\beta^2\gamma-\alpha^2|/\alpha^2 = (\alpha^2-\alpha\beta^2\gamma)/\alpha^2 = (\alpha-\beta^2\gamma)/\alpha.$$

Case 1(c)(−): $p < 0$, $\Delta < 0$, and $\alpha^2 - \alpha\beta^2\gamma \le 0$.
Then $-3\alpha^2 \ge -\alpha^2 - 2\alpha\beta^2\gamma$ and therefore $x_+ = -1$. It follows that our desired root is the "largest root" corresponding to $k = 0$ in Theorem 4.1(i)(c):

$$\begin{aligned} x &= x_0 + 2(-p/3)^{1/2} \cos\Big(\frac{1}{3}\Big(\arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} + \frac{2|\beta^2\gamma-\alpha|}{3|\alpha|} \cos\Big(\frac{1}{3}\Big(\arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big) \\ &= -\frac{2\alpha+\beta^2\gamma}{3\alpha} + \frac{2(\beta^2\gamma-\alpha)}{3\alpha} \cos\Big(\frac{1}{3}\Big(\arccos\frac{-q/2}{(-p/3)^{3/2}}\Big)\Big), \end{aligned}$$

where in the last line we used the assumption to deduce that

$$|\beta^2\gamma-\alpha|/|\alpha| = |\alpha\beta^2\gamma-\alpha^2|/\alpha^2 = (\alpha\beta^2\gamma-\alpha^2)/\alpha^2 = (\beta^2\gamma-\alpha)/\alpha.$$

Note that the last two cases can be combined to obtain (71a).

Case 2: $p = 0$, i.e., $\alpha - \beta^2\gamma = 0$ by (68).
Then $\Delta = (q/2)^2 \ge 0$; hence, $\sqrt{\Delta} = |q|/2$ and thus $\{-q/2 \pm \sqrt{\Delta}\} = \{-q, 0\}$. By Theorem 4.1(ii), the only real root is

$$x_0 + (-q)^{1/3} = x_0 + (-q/2+\sqrt{\Delta})^{1/3} + (-q/2-\sqrt{\Delta})^{1/3}$$

which is the same as (70) using (110).

Case 3: $p > 0$. In view of (68), this case never occurs and we are done.

F. Proof of Example 9.1

The first cases were already provided in Example 24.57 of [3].
Now assume $\|y\|^2 + 2\gamma\eta > 0$. It was also observed in this Example 24.57 that if $y = 0$, we then have $\mathrm{Prox}_{\gamma h}(y,\eta) = (0,\eta)$.
So assume also that $y \ne 0$. It follows from the discussion that in [3, Example 24.57] that λ is the unique positive solution of the already depressed cubic

$$\lambda^3 + \frac{2(\eta+\gamma)}{\gamma}\lambda - \frac{2\|y\|}{\gamma} = 0, \tag{115}$$

which is where the discussion in [3] halted. Continuing here, we set

$$p := \frac{2(\eta+\gamma)}{\gamma}, \quad q := -\frac{2\|y\|}{\gamma} < 0, \tag{116}$$

and $\Delta := (p/3)^3 + (q/2)^2$.

Using Corollary 3.2, we see that if $\Delta < 0$, then

$$\lambda = 2(-p/3)^{1/2} \cos\left(\frac{1}{3} \arccos \frac{-q/2}{(-p/3)^{3/2}}\right) \tag{117}$$

while if $\Delta \geq 0$, then

$$\lambda = \sqrt[3]{\frac{-q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}} \tag{118}$$

which slightly simplifies to the expression provided in (75).

Finally, notice that if $y = 0$, then the assumption that $\|y\|^2 + 2\gamma\eta > 0$ yields $\eta > 0$; thus, $p > 0$, $q = 0$, and hence $\Delta > 0$. Formally, our λ then simplifies to 0 which conveniently allows us to combine this case with the case $y \neq 0$. The proof is complete.

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